

proposed, and so learning conservation could not therefore follow from logical analysis of the domain. Instead, conservation is an empirical fact about the world and turns crucially on the notion of “relevance” of the transformation. A favorite teaching trick of mine after a class on conservation, is to ask students whether the area bounded by a piece of string is changed upon altering the shape of the string slightly. Most students will say “no” and, of course, will be shown to be wrong as the string is flattened into a shape with bounded area of zero. The point is made more forcefully if you then invite a 3-year-old into the room and they get the correct answer!

If conservation is learned empirically rather than logically, then figuring out that substances or individuals do not change quantity upon being poured into a different container or spatially rearranged, would involve learning about the nature of matter and objects in the physical world. It isn't so much experience with objects in counting situations that bootstraps these number principles, nor that these principles are arrived at autonomously within the unconscious theorem-generating mental number space, but that the indirect process of trying to figure out the nature of objects as objects and individuals within sets might be the catalyst for arriving at these principles in the relatively informal manner that we also acquire knowledge of other domains of folk-science. The inference that these principles apply universally to all quantities or numbers, might be understood at the folk-math level in the same way that we have faith that the sun will rise tomorrow, by everyday Hume-style, deductively invalid, inductive generalization. The formal use of successor functions and mathematical induction/deduction might only come into play when children are taught explicitly in school.

Set representations required for the acquisition of the “natural number” concept

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Abstract: Rips et al. consider whether representations of individual objects or analog magnitudes are building blocks for the concept NATURAL NUMBER. We argue for a third core capacity – the ability to bind representations of individuals into sets. However, even with this addition to the list of starting materials, we agree that a significant acquisition story is needed to capture natural number.

Rips et al. paint a picture of math knowledge as divorced from the object tracking and simple enumeration abilities of young children. Something is right here, and something is wrong. What is right is that arithmetic is a domain of relations across numbers, numbers (at least on many stories) are abstract objects, and the truths of mathematics are not dependent on anything in the world, like objects or approximation abilities. But what is wrong with this picture is that neither humans over the course of history nor children over the course of development initially understand mathematics as a domain concerning abstract entities divorced from the world. The ability to describe relationships across real-world collections of objects was presumably the historical impetus for discovering arithmetic, it is how children begin engaging in mathematics, and for most humans, it remains the major forum for applying arithmetic knowledge. *A fortiori*, a bridging mechanism between mathematical truths about numbers (i.e., abstract objects) and collections (i.e.,

things in the world) is required if such knowledge is to be made relevant to real-world calculation.

We suggest that the concept set and ordered relations across sets can serve as a bridging mechanism (another possibility is plural variables). Rips and colleagues do not list set representations among the early capacities potentially relevant for math knowledge. We suggest that the concept set is required and that this notion cannot come from object tracking, the approximate number system, or language.

Conceiving of a set requires representing the hierarchical relationship between individual items and the larger structure into which they are bound (note that this is distinct from perceptual grouping or recoding, both processes that destroy the individuals comprising the group). Whereas individuals exist in the world, sets are abstractions that exist in the mind; it takes a mind to maintain multiple hierarchically ordered construals.

The object tracking system (and working memory) will not provide the concept set. This system will not return a representation of one-ness, two-ness, or three-ness when it is engaged in tracking one, two, or three objects. This is because the objects are completely separate individuals in the world; maintaining this separateness was one of the main motivators for positing a tracking system in the first place.

Nor will the approximate number system (ANS), which produces analog magnitude representations, provide the concept set. The representation of 1 in the ANS is a continuous distribution of activation (e.g., Gallistel and Gelman [2000] describe this system as instantiating the real numbers, not the integers). Also, the representation for 8 in the ANS (the 8-distribution) is in no way made up of 8 individual distributions of 1. Further, on some models, the entities (in the world) that enter into this representation are not even individuated (in the mind) prior to creating this estimate (rather, they are Gaussian activations passed through a continuous normalizing filter; Dehaene & Changeux 1993). With no discrete representation of 1, no compositionality, and, on some models, no individuation prior to enumeration, the ANS cannot provide the concept set.

Nor can language do the job. “SET” is a notion with content. The syntax provides algorithms for manipulating content. Although the language faculty might provide relevant computational possibilities, including embedding, agreement, and recursion, these are all content-free syntactic operations. One must maintain a distinction between the computation that binds any two individuals into a set and the concept of a set of two individuals. The former might reside in the language faculty, but not the latter, and it is the latter that is needed to bridge math to the world.

What then is needed to connect set with number? If we begin with object tracking, which will provide us with representations of discrete and separate individuals, one might take these separate individuals (e.g., two separate object files) and bind them into a set of individuals. Rather than tracking, for example, Joe and Jenny, one could then index the set of Joe and Jenny. Our recent work suggests that 14-month-old infants make this distinction between tracking individuals and tracking sets of individuals (Feigenson & Halberda 2004; in press). Importantly, this work provides evidence that infants maintain representations of the individuals comprising the set, thereby distinguishing set representations from demonstrations of perceptual grouping and tracking (e.g., Wynn et al. 2002), in which there is no evidence that infants retain representations of a group's components.

Next, the learner must abstract away from any particular instance of tracking, for example, a set of two particular individuals, to represent the more general case of tracking a set of any two individuals. Presently, we know of no evidence that infants construct this more general representation, but we see no computational limitation that makes this abstraction impossible. Last, the learner must order set representations with, for example, 1, 2, and 3 individuals according to the successor

function. With this ordering, set representations become bona fide representations of one-ness, two-ness, and three-ness (modulo Rips et al.'s comments concerning mod_{10}). This ordering appears to occur in the course of learning to enumerate collections.

The set-binding computation we discuss may require that items be represented in parallel prior to binding. Limits on parallel individuation or on working memory would therefore limit these set representations to smaller numbers of items. Our guess is that precise large numbers will be known only through a meta-language, or schemas along the lines Rips et al. suggest. Large numbers may derive their semantic content via the functional role they play in relations with other numbers, but ultimately they make contact with the smaller cardinalities, for which set representations are also available.

We suggest that the concept SET will be a requirement for any theorist who tries to build an acquisition story from early enumeration abilities to NATURAL NUMBER. Furthermore, any story that divorces natural number from these early abilities will require the concept SET (or some other bridge), applicable to real-world collections, in order to connect purely mathematical knowledge to the real world.

Recursive reminding and children's concepts of number

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Abstract: According to the recursive reminding hypothesis, repetition interacts with episodic memory to produce memory representations that encode – and recursively embed – experiences of reminding. These representations provide the rememberer with a basis for differentiating among the first time something happens, the second time it happens, and so on. I argue that such representations could mediate children's understanding of natural number.

If Rips et al.'s target article is right, children's understanding of natural number cannot be derived from numerical abilities that they displayed as infants. Two crucial elements that are missing in the article are a basis for the concept of successor and a schema for the starting value of one. I suggest here that ideas from the field of memory may help to fill in these gaps. Specifically, I propose that everyone with a functioning episodic memory possesses a pre-symbolic analogue of counting, which could mediate the acquisition of number concepts.

To start in the memory laboratory, consider an experiment in which the items to be enumerated are not simultaneously displayed in front of the subject, but are dispersed over time. A standard experiment might use 50 different items (e.g., words or pictures), presented in a list totaling 150 study trials. Different stimuli are presented different numbers of times, haphazardly interleaved with presentations of the other items. After the list has ended, subjects are shown each item and asked how many times it appeared. People are surprisingly good at making such frequency judgments. Telling subjects to attend to presentation frequency ahead of time has little or no effect on performance, and 4-year-olds judge frequency about as well as adults (Hasher & Chromiak 1977). "Memory strength" does not appear to be the basis of this ability, because one cannot trick subjects into saying that a once-occurring item appeared two times merely by presenting it recently (Wells 1974) or for a long study duration (Hintzman 2004). It is not plausible that subjects are counting, in the strict sense of the term, because this would require maintaining running totals for many different

items, and the usual study instructions give subjects no reason to count.

How can we understand such findings? Hasher and Zacks (1979) proposed that humans are specially prepared by evolution to acquire frequency information. A different hypothesis (Hintzman 2004) is that the ability is not innate, as such, but derives directly from the operation of more basic memory processes, which yield recursive representations of repeated experience.

I assume that everyone with a functioning episodic memory has three basic abilities: (1) to maintain a complex mental state, (2) to encode a representation of that state into memory, and (3) to retrieve the representation reflexively, when given a matching retrieval cue. I also assume that the mental state that results from retrieval has two parts: one that refers to the present moment (i.e., the encounter with the cue) and one that refers to the past (the reminding of an earlier experience). This distinction between present and past is a fundamental aspect of the rememberer's subjective experience. Schematically, if A represents the subject's mental state upon first encountering a particular stimulus, then the mental state upon encountering the stimulus for the second time is $A + R(A)$.

A crucial consequence of the second encounter with the stimulus is that the corresponding experience, $A + R(A)$, is itself encoded in memory. Now if the same stimulus is encountered yet again, the person is reminded of having been reminded on the previous trial, so the resulting mental state is $A + R[A + R(A)]$, which is itself encoded in memory, and so on. In this notation, the R outside brackets is the current reminding, and the R inside brackets is the reminding that is remembered from the previous trial. It is important to understand that according to the hypothesis, the memory representation itself is recursive, not just the process that generates the representation. Each new mental state – and therefore each new encoding – is a compound that may include (or refer to) previous mental states. It appears that assumptions (1) to (3) stated above will automatically generate recursive representations. A theorist who adopts those assumptions but does not want recursive representations would have to include a restriction that specifically rules them out.

I assume that on a frequency judgment test, the subject retrieves the test item's latest memory representation and maps its depth of embedding or recursion onto the number scale (Hintzman 2004). There may be no limit to the number of levels of reminding that a memory can represent. Frequency judgments become less and less accurate in an absolute sense as frequency goes higher, but there is no evidence that they ever level off. Certainly, memory mechanisms are subject to several kinds of "noise," so if there is a memory limit, it may be more akin to a limit of resolution imposed on the entire representation than to a strict limit on the number of levels of recursion that can be encoded in a memory trace.

Of course, a child seldom encounters a series of events as uniform as a list of items that includes exact repetitions. The child does, however, have many novel or first-time experiences, and many experiences that are repetitive or routine. I assume that most experiences of the latter type include reminders of similar experiences from the past. So, if the hypothesis of recursive reminding is applicable to list learning experiments, it may also be applicable to a child's everyday life. The only precondition is a functioning episodic memory. With iterative application of encoding and retrieval, recursive representations will essentially build themselves.

How could this help the child acquire the concepts of one and successor? Let us suppose that all reminders possess a common element, R, which is tacitly understood to reference the past. The mental states $A + R(A)$ and $B + R(B)$, are similar because they include this common element. By way of contrast, the mental states A and B are similar in that the element of reminding is absent. Children know what it means to remember, and what it means to not remember, before the age of three (Fivush &