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A dissociation between small and large numbers in young children's ability to "solve for x" in non-symbolic math problems

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ABSTRACT

Solving for an unknown addend in problems like 5 + x = 17 is challenging for children. Yet, previous work (Kibbe & Feigenson, 2015) found that even before formal math education, young children, aged 4- to 6-years, succeeded when problems were presented using non-symbolic collections of objects rather than symbolic digits. This reveals that the Approximate Number System (ANS) can support pre-algebraic intuitions. Here, we asked whether children also could intuitively "solve for x" when problems contained arrays of four or fewer objects that encouraged representations of individual objects instead of ANS representations. In Experiment 1, we first confirmed that children could solve for an unknown addend with larger quantities, using the ANS. Next, in Experiment 2a, we presented addend-unknown problems containing arrays of four or fewer objects (e.g., 1 + x = 3). This time, despite the identical task conditions, children were unable to solve for the unknown addend. In Experiment 2b, we replicated this failure with a new sample of children. Finally, in Experiment 3, we confirmed that children's failures in Experiments 2a and b were not due to lack of motivation to compute with small arrays, or to the discriminability of the quantities used: children succeeded at solving for an unknown *sum* with arrays containing four or fewer objects. Together, these results suggest that children's ability to intuitively solve for an unknown addend may be limited to problems that can be represented using the ANS.

1. Introduction

Children are introduced to formal mathematics starting in early elementary school, but the process of acquiring formal mathematical skills is protracted. One reason formal math is thought to be particularly challenging for children is that it requires young learners to mentally manipulate symbols according to a set of learned rules (Kieran, 1992; Nathan, 2012; Susac, Bubic, Vrbanc, & Planinic, 2014; Van Amerom, 2003). For example, a child encountering the problem $2 + 3 = x$ must understand the meanings of the symbols (digits and operators) and the algorithm for combining the two digits as specified by the operator symbol. Misunderstanding of or difficulty processing the meanings of mathematical symbols predicts poorer mathematical performance in children (Byrd, McNeil, Chesney, & Matthews, 2015; Desoete, Ceulemans, De Weerdt, & Pieters, 2012). And as mathematics becomes more complex over successive years of instruction, requiring the manipulation of variables as well as digits and operators, learners continue to struggle even into the college years (Koedinger, Alibali, & Nathan, 2008).

Although learning to manipulate the symbols used in formal mathematics is challenging, infants, children, adults, and non-human animals have fundamental mathematical intuitions that do not depend on external symbols. These populations all share an Approximate Number System (ANS) that allows them to estimate the number of items in visual and auditory arrays without language, education, or symbolic notation (e.g., Dehaene, 1997; Feigenson, Dehaene, & Spelke, 2004; Libertus & Brannon, 2009). Unlike the exact number representations involved in most symbolic math, the number representations generated by the ANS are noisy and imprecise—this remains true throughout the lifespan, even after children have learned to represent exact number symbolically (Carey, 2009; Halberda & Feigenson, 2008a; Halberda, Ly, Wilmer, Naiman, & Germine, 2012; Xu & Spelke, 2000).

The nature of the relationship between the ANS and acquired school mathematical abilities remains the topic of much debate. However, evidence suggests that the ANS plays a role in school math achievement, despite most of school mathematics requiring the kinds of precise representations that the ANS lacks. First, individual differences in the precision of the ANS correlate with and

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predict symbolic math performance (e.g., Chen & Li, 2014; DeWind & Brannon, 2012; Feigenson, Libertus, & Halberda, 2013; Gilmore, McCarthy, & Spelke, 2010; Halberda, Mazzocco, & Feigenson, 2008; Halberda et al., 2012; Libertus, Feigenson, & Halberda, 2011; Starr, Libertus, & Brannon, 2013, but see also, e.g., Holloway & Ansari, 2009; Iuculano, Tang, Hall, & Butterworth, 2008; Soltesz, Szucs, & Szucs, 2010). Second, training of numerical approximation abilities has been found to improve symbolic math performance in adults and children (Hyde, Khanum, & Spelke, 2014; Park, Bermudez, Roberts, & Brannon, 2016; Park & Brannon, 2013, 2014; Wang, Odic, Halberda, & Feigenson, 2016).

One way in which the ANS might be useful during the process of initially learning formal mathematics is by providing basic intuitions about numerical computations. Indeed, despite their noisiness, ANS representations can support many of the computations that are later encountered in formal mathematics, including ordering (Lipton & Spelke, 2005), addition and subtraction (Barth et al., 2006; Booth & Siegler, 2008; Gilmore, McCarthy, & Spelke, 2007; McCrink & Wynn, 2004), multiplication (McCrink & Spelke, 2010), and division (McCrink & Spelke, 2016). Critically, recent research suggests that presenting problems non-symbolically, using arrays that encourage the use of ANS representations, can help children solve at least some of the more complex computations that are used in formal schooling – even problems that many children struggle with into adolescence. In this previous work (Kibbe & Feigenson, 2015), we found that, not surprisingly, 4- to 6-year-old children failed to solve symbolically presented pre-algebraic problems (i.e., problems that required solving for an unknown addend, like "6 + x = 18," presented using digits). Yet these children spontaneously "solved for x" when the very same problems were presented non-symbolically using collections of objects. In these studies, children were introduced to a "magic cup" that always transformed object collections by a constant quantity. Then they saw a starting quantity (e.g., six objects), watched as the magic cup was applied to that quantity, and finally saw a new quantity (e.g., 18 objects) revealed. After seeing events like this, children were able to correctly infer the approximate quantity in the magic cup—in this sense, they solved for the value of the unknown addend *x*.

These results suggest that presenting problems non-symbolically, with collections of objects instead of written or spoken number symbols, can sometimes help children perform specific mathematical computations earlier than they otherwise could. Harnessing ANS representations appears to allow children to form "gut-sense" estimates of the quantities involved, even when the quantities' values had to be inferred. However, ANS representations are limited in some important ways. Whereas symbolically mediated exact number representations allow children to form very precise representations of x in a "solve for x" task, ANS representations inherently provide only noisy estimates. These estimates were sufficiently precise to allow children to succeed in our "magic cup" task – for example, after seeing six buttons transformed by the magic cup to yield 18 buttons, children correctly identified the cup as containing 12 buttons rather than 4 or 24. Distinguishing between 4 and 12, or 12 and 24, can be accomplished even from noisy estimates. But ANS representations, because of their inherent imprecision, should not support discrimination of the target from numerically nearer distractors (e.g., 12 versus 13 buttons).

ANS representations may pose yet a further limitation on children's ability to solve for unknown variables. Much evidence suggests that whereas ANS representations are readily deployed in response to large quantities (usually quantities greater than three), they often fail to be deployed in response to smaller quantities. Instead, young children presented with one, two, or three items often appear to represent these arrays in terms of individual objects (Object A, Object B, Object C) rather than as a single entity with an approximate (or exact) cardinality (Coubart, Izard, Spelke, Marie, & Streri, 2014; Feigenson & Carey, 2003, 2005; Feigenson, Carey, & Hauser, 2002; Feigenson, Carey, & Spelke, 2002; Hyde & Spelke, 2011; vanMarle, 2013; Xu, 2003). Although under some circumstances infants can be induced to represent arrays of one, two, or three objects using approximate number representations (e.g., Cordes & Brannon, 2009), small and large arrays often appear to trigger the deployment of two separate representational systems. An open question, then, is whether children can solve for the value of an unknown variable using individual object representations rather than approximate number (ANS) representations. If children can "solve for x" with small quantities as well as large ones, this would suggest that pre-algebraic computations can be performed over multiple types of quantity-relevant representations, as long as external symbols (digits or words) are not required.

Here we tested this possibility by contrasting children's ability to non-symbolically "solve for x" with large versus small numbers of objects. We tested children of the same age as in our previous study, using the same non-symbolic "magic cup" task (Kibbe & Feigenson, 2015). First, in Experiment 1, we sought to replicate children's success at solving for the value of an unknown addend when the quantities involved large numerosities only. Next, in Experiment 2a, we asked whether children also could solve for x with small quantities of four or fewer – quantities that have been found by previous work to activate the system for representing individual objects rather than approximate cardinalities. To preview, we found that children succeeded in Experiment 1, but failed in Experiment 2a. In Experiment 2b, we replicated children's failure to solve for x with small quantities with a separate sample of children. Finally, in Experiment 3, we asked whether children's failures in Experiments 2a and 2b were due to representing transformations over small quantities, versus performing pre-algebraic computations. We found that when children were asked to solve for the value of an unknown sum, rather an unknown addend, they succeeded. We close by discussing the implications of these results for our understanding of children's early numerical abilities.

2. Experiment 1: Unknown addend, large quantities

The purpose of Experiment 1 was to replicate the finding that 4- to 6-year old children can solve for x when presented with non-symbolic arrays containing large numbers of objects. Children were introduced to a magic cup and were told that this cup always added the same number of objects to an existing collection. Children then saw the magic cup demonstrated on three different starting quantities (i.e., the cup added *x* to three different starting arrays). Finally, children were asked to choose which of two non-symbolic quantities the magic cup contained – i.e., they were asked to solve for *x*.

2.1. Participants

Twenty-four children (mean age: 5 years, 6 months; range: 4 years 1 month – 6 years 11 months; 10 girls) participated in the children's wing of a local science museum. Children received a sticker for their participation.

2.2. Methods

2.2.1. Materials

Materials consisted of a small stuffed alligator toy and a 10-oz white paper cup. The cup could transform the quantity of three different types of arrays: buttons, pennies, and small toy shoes. A second 10-oz white paper cup and a set of pink and purple pom-poms were used during the Test Trial. All objects measured between 0.75 and 2 cm.

2.2.2. Procedure

2.2.2.1. Demonstration trials. Children sat across from the experimenter at a child-sized table in a quiet corner of the museum. The experimenter first showed children the stuffed animal, Gator, and the white paper cup, which was placed upside down on the table. The experimenter told children that Gator had a "magic cup" and that, "No matter what, if I put a pile of things in front of Gator, his magic cup will come and add more things to the pile. And it's always going to add the same number of things every time no matter what. Want to see how it works?" In Demonstration Trial 1, the experimenter placed five buttons on the table, tightly clustered together in order to discourage children from attempting to count the objects individually. She then pointed to the pile and said, "See these buttons?" After about 5 s, she said, "Now watch carefully, here comes the magic cup!" She then completely covered the buttons with the inverted cup, shook the cup while keeping it pressed against the table surface, and then lifted it to reveal 17 buttons. She then said, "Did it work? Yeah, it worked! See Gator's buttons now?" Children viewed the new, larger array of buttons for 5 s, after which the experimenter cleared all of the objects and the cup off the table. Fig. 1 shows a schematic of this Demonstration Trial.

The experimenter then asked children if they thought the magic cup would work on other types of objects, and conducted two more Demonstration Trials using the same procedure with different objects (see Table 1). In the second Demonstration Trial, the magic cup transformed a pile of nine pennies into 21 pennies; in the third, the magic cup transformed a pile of six toy shoes into 18 shoes. Across the three Demonstration Trials, different initial quantities were used but the magic cup always added the same quantity (12). Crucially, children never got to see what was in the cup; they only saw the initial quantity and the final quantity. Thus, the contents of the cup acted as an unknown addend.

2.2.2.2. Test trial. In the single Test Trial, the experimenter asked children whether they thought the magic cup would work on pom-poms. The experimenter pretended to look for Gator's magic cup under the table. She then placed two identical cups upside down on the empty table, 60 cm apart, and said, "Uh-oh, I found two magic cups under the table, but I don't know which one is Gator's! I am going to show you what's inside, and you can tell me which one is Gator's magic cup." She then lifted both cups simultaneously to reveal two different quantities of pom-poms (chosen because these were new objects that children had not yet encountered in the Demonstration Trials): the target quantity (12) and a distractor quantity (either 4 or 24, counterbalanced across participants) (Fig. 1). She then asked children, "Which cup is Gator's?" Children were given the opportunity to point to one of the two cups. Regardless of children's choice, the experimenter said "Great job!" Whether the target quantity was presented on the left or the right was counterbalanced across participants.

2.2.2.3. Quantities. The quantities used in Experiment 1 were identical to those in our previous study (Kibbe & Feigenson, 2015; Table 1). Across the three Demonstration Trials, Gator's cup added 12 to piles of five, nine, and six objects. These starting quantities were chosen to be outside the subitizing range in which items can be represented individually (Gelman, 1977; Trick & Pylyshyn, 1994) and to be sufficiently discriminable from the quantity being

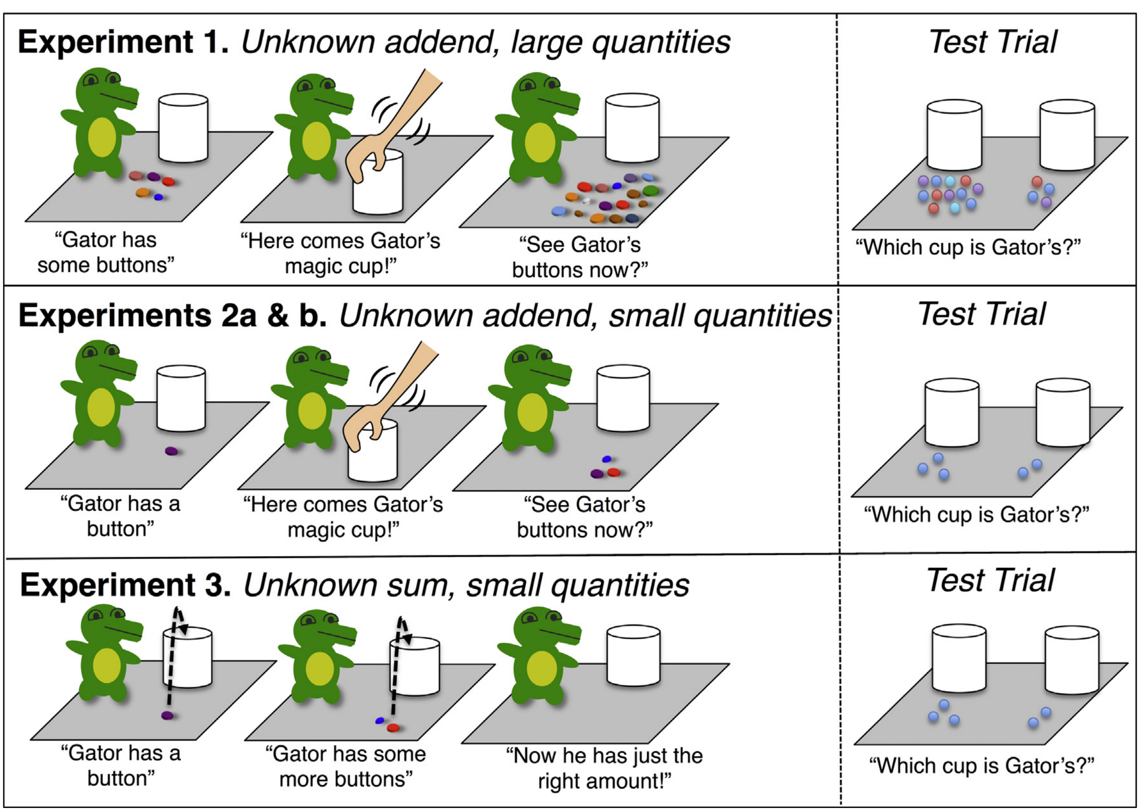

Fig. 1. Left panels present schematics of the first of three Demonstration Trials in Experiments 1, 2a, 2b, and 3. In Experiments 1, 2a, and 2b, children saw a starting quantity, saw the magic cup add an unknown quantity to this starting quantity, and then saw the new, transformed quantity. In Experiment 3, children saw two quantities added separately to Gator's cup, such that the sum was unknown. Right panels show the single Test Trial, in which children chose which of two quantities was in Gator's cup.

Table 1
Quantities used and children's performance in Experiments 1, 2a, 2b, and 3.

	Demonstration trials	Value of x	Test trial choice	% Correct
Experiment 1	5 + x = 17 (buttons)	12	4 vs. 12	83.3%
Unknown addend	9 + x = 21 (pennies)		*or*	$p = 0.002$
	6 + x = 18 (shoes)		12 vs. 24	
Experiment 2a	1 + x = 3 (buttons)	2	1 vs. 2	54.2%
Unknown addend	2 + x = 4 (pennies)		*or*	$p = 0.84$
	1 + x = 3 (shoes)		2 vs. 3	
Experiment 2b	1 + x = 3 (buttons)	2	1 vs. 2	56.3%
Unknown addend	2 + x = 4 (pennies)		*or*	$p = 0.59$
	1 + x = 3 (shoes)		2 vs. 3	
Experiment 3	1 + 2 = x (buttons)	3	2 vs. 3	83.3%
Unknown sum	2 + 1 = x (pennies)		*or*	$p = 0.002$
	1 + 2 = x (shoes)		3 vs. 4	
	or			
	1 + 1 = x (buttons)	2	1 vs. 2	
	1 + 1 = x (pennies)		*or*	
	1 + 1 = x (shoes)		2 vs. 3	

added by the magic cup (Halberda & Feigenson, 2008a, 2008b). In the Test Trial, the distractor quantities also were chosen to be sufficiently discriminable from the target quantity, but also to be somewhat close to the starting and ending quantities in the Demonstration trials, so as to minimize perceptual novelty.

2.3. Results

When asked which of two visible quantities was the quantity inferred to be in Gator's cup, 20/24 children correctly chose the target quantity (83.3%; chance = 50%, binomial test $p = 0.002$, two-tailed). We found no significant difference in children's performance as a function of the distractor quantity, although more children succeeded when the distractor cup contained four objects than when it contained 24 (12/12 children succeeded when the distractor cup contained four; 8/12 succeeded when the distractor cup contained 24; Fisher's exact test $p = 0.09$). There also was no significant difference in the performance of girls versus boys (10/10 girls and 10/14 boys succeeded; Fisher's exact test $p = 0.11$). Finally, to assess whether age was a factor in children's performance, we ran a one-way ANOVA on participants' ages (in days) with their response (correct versus incorrect) as a between-subjects factor. We found no effect of age ($F_{1,22} = 2.24$, $p = 0.15$). This result was confirmed non-parametrically using the Freeman-Halton extension of Fisher's exact test for a 3 (age: 4 year-olds, 5 year-olds, 6 year-olds) × 2 (correct or incorrect) contingency table ($p = 0.22$).

2.4. Discussion

Experiment 1 replicated our previous findings (Kibbe & Feigenson, 2015), confirming that 4 to 6 year-old children can infer the approximate quantity of an unknown addend. After seeing addend-unknown problems that were instantiated non-symbolically using large collections of objects, children successfully inferred the value of the unknown addend. Given that in our previous work children failed to solve the same problems when presented symbolically (Kibbe & Feigenson, 2015), and given that children use ANS representations to intuitively solve other arithmetic problems involving unknown sums, differences, products, and quotients (Barth, Baron, Spelke, & Carey, 2009; Barth, LaMont, Lipton, & Spelke, 2005; Barth et al., 2006; McCrink & Spelke, 2010, 2016), it seems likely that children relied on representations from the ANS in this task.

However, this result leaves open the question of whether children can flexibly use multiple types of quantity-relevant representations to solve pre-algebraic problems. In Experiment 2a, we asked whether children could infer the value of an unknown addend with arrays containing four and fewer objects—arrays that have been shown to trigger the system for representing individual objects, rather than approximate cardinalities (see Feigenson et al., 2004 for review). Children saw a series of Demonstration Trials identical to those in Experiment 1, except that Gator's magic cup added two objects to starting arrays of one, two, and one objects ($x = 2$). In the single Test Trial, children again were presented with two cups, one containing the target quantity of two and the other containing a distractor quantity of one or three, and were asked to choose which magic cup belonged to Gator.

3. Experiment 2a: Unknown addend, small quantities

3.1. Participants

Twenty-four children (mean age: 5 years, 7 months; range: 4 years 4 months to 6 years 10 months; 12 girls) participated at the science museum.

3.2. Methods

3.2.1. Materials and procedure

Materials were identical to Experiment 1. The procedure was identical to that of Experiment 1, except that different quantities were used (Table 1). In the three Demonstration Trials, children saw Gator's magic cup transform arrays of one button, two pennies, and one shoe into arrays of three buttons, four pennies, and three shoes, respectively. During the single Test Trial, the experimenter again pretended to be confused about which of two identical magic cups was Gator's. One cup was shown to contain the target quantity (two pom-poms) and the other contained a distractor quantity (either one or three pom-poms), and children were asked to choose which of these was Gator's magic cup.

3.2.2. Quantities

The small quantities used in Experiment 2a were chosen to be within the range shown by previous research to frequently trigger individual object file representations (see e.g., Feigenson et al., 2004; Hyde, 2011). In the Test Trial, the distractor quantities were chosen to be familiar (i.e., they had been seen as starting or ending quantities during one of the other two Demonstration trials) so that children would not simply avoid choosing distractor quantities on the basis of novelty. We only presented target and distractor quantities that differed by a ratio shown to be discriminable even by much younger infants (e.g., Feigenson, Carey, & Hauser, 2002; Starkey & Cooper, 1980).

3.3. Results

When asked to choose which of two visible quantities they inferred to be in Gator's cup, only 13/24 children correctly chose the target quantity (54.2%; chance = 50%, binomial test $p = 0.84$, two-tailed). There was no difference in girls' and boys' performance (6/12 girls and 7/12 boys succeeded; Fisher's exact test $p = 1.0$) and no effect of age (one-way ANOVA $F_{1,22} = 2.09$, $p = 0.16$; Fisher-Freeman-Halton exact test $p = 0.68$). However, we found a significant difference in children's performance as a function of the quantity in the distractor cup, with 10/13 children successfully avoiding the distractor cup when it contained one object and only 3/11 successfully avoiding the distractor cup when it contained three (Fisher's exact test $p = 0.04$). Hence children tended to choose the larger of the two test quantities, regardless of which was the correct target.

The design of Experiment 2 meant that in the Test Trial, children were asked to discriminate ratios of 1:2 (when the distractor quantity was one) or 2:3 (when the distractor quantity was three), whereas children in Experiment 1 were asked to discriminate ratios of 1:2 (when the distractor quantity was 24) or 1:3 (when the distractor quantity was 4). This leaves open the possibility that children's failure in Experiment 2a was due to the children who were tested with the harder 2:3 ratio. To investigate this possibility, we analyzed data from 12 additional children (mean age: 5 years 5 months, range: 4 years 1 month – 8 years 0 months; 4 girls) who saw three Demonstration Trials in which Gator's magic cup added 12 objects to arrays of 5, 9, and 6 objects, just as in Experiment 1. In the Test Trial, these children were asked to choose between the target (12) and a distractor quantity of 18, a 2:3 ratio. We found that 9/12 children (75%) succeeded. Performance was not significantly different from that in Experiment 1, in which children were tested with larger numerosities with a 1:2 test ratio (Fisher's exact test $p = 1.0$), but it was significantly different from performance in Experiment 2a, in which children were tested with small numerosities with a 2:3 ratio (Fisher's exact test $p = 0.04$).

3.4. Discussion

Children in Experiment 2a failed to "solve for x," whereas children in Experiment 1 succeeded. The experiments only differed in the quantities used: in Experiment 1, the quantities were relatively large (range: 4–21 objects) and therefore likely to be represented by the ANS, whereas in Experiment 2a the quantities were small (range: 1–4 objects) and therefore likely to be represented using individual object representations. This divergence in performance was not caused by differences in the ratio between the target and distractor quantity, as children succeeded with large numerosities and failed with small ones even with identical ratios. Instead, our results suggest that children have difficulty inferring the quantity of an unknown addend when presented with arrays in the small number range.

The failure of children in Experiment 2a is especially surprising given that children of this age can count reliably in the small number range (e.g., LeCorre & Carey, 2007), and therefore could plausibly have used verbally mediated representations of exact number (counting) to solve the task. Children's failure despite the seemingly less demanding nature of the numerical problems presented in Experiment 2a (as compared with Experiment 1) led us to attempt to directly replicate our findings in a separate experiment with a larger sample of children.

4. Experiment 2b: Unknown addend, small quantities (replication of Experiment 2a)

4.1. Participants

Thirty-two children (mean age: 5 years, 1 month; range: 4 years 1 months to 6 years 9 months; 19 girls) participated at the science museum.

4.2. Materials and procedure

The materials and procedure were identical to those in Experiment 2a (see also Table 1).

4.3. Results

When asked to choose which of two visible quantities was inferred to be in Gator's cup, 18/32 children correctly chose the target quantity (56.3%; chance = 50%, binomial test $p = 0.59$, two-tailed). There was no difference in girls' and boys' performance (11/19 girls and 7/13 boys succeeded; Fisher's exact test $p = 1.0$) and no effect of age (one-way ANOVA $F_{1,30} = 1.59$, $p = 0.22$; Fisher-Freeman-Halton exact test $p = 0.78$). We also found no difference in children's performance as a function of the quantity in the distractor cup; slightly more children successfully avoided the distractor cup when it contained one object (10/16 children) than when the distractor cup contained three (8/16 children), but this difference was not significant (Fisher's exact test $p = 0.72$).

We found no difference in children's performance in Experiments 2a and 2b (Fisher's exact test $p = 1.0$, two-tailed). In contrast, there was a significant difference in children's performance in Experiment 1 versus children's combined performance in Experiments 2a and 2b (Fisher's exact test $p = 0.02$, two-tailed).

Finally, we conducted a Bayes Factor analysis on children's performance in Experiments 2a and 2b. While traditional null hypothesis significance testing only allows us to either reject or fail to reject the null hypothesis, Bayes Factor analysis allows us to obtain statistical support *for* the null hypothesis (Rouder, Speckman, Sun, Morey, & Iverson, 2009) by providing the odds that the data were generated by a binomial process with probability 0.5. Because the null hypothesis – that children cannot solve for an unknown addend when problems are presented with arrays of four or fewer objects – is of theoretical importance, obtaining statistical odds for the null hypothesis would allow us to accept the null hypothesis with confidence. A Bayes Factor of 3 or greater is roughly equivalent to the $p = 0.05$ significance level in traditional statistics (Gallistel, 2009). Bayes Factor analysis conducted on children's overall performance yielded odds of 3.72:1 in favor of the null hypothesis in Experiment 2a, and odds of 3.62:1 in favor of the null hypothesis in Experiment 2b. Conversely, Bayes Factor analysis conducted on children's performance in Experiment 1 yielded odds of 63.16:1 *against* the null hypothesis.

4.4. Discussion

In Experiment 2b, we replicated children's surprising failure to "solve for x" with quantities of four or fewer. The results of Experiments 1, 2a, and 2b suggest that the computations that enable children to infer the quantity of an unknown addend from nonsymbolic arrays readily operate over ANS representations, but do not readily operate over representations of individual objects.

However, an alternative possibility is that ancillary task demands might have led children to fail to solve for *x* with small quantities. For example, children might not have been motivated to track the small quantities presented in Experiments 2a and 2b, perhaps because they did not find the smaller arrays to be visually salient. If so, then children might have failed to perform *any* mental transformations over arrays of small objects, rather than specifically failing to infer the value of an unknown variable. An additional concern is that children might have failed in Experiments 2a and 2b because, for half of the children, the choice between the target quantity and the distractor quantity instantiated a smaller, less discriminable ratio difference (2:3). Indeed, children in Experiments 2a and 2b were more successful at choosing the target quantity when the distractor quantity was one than when it was three. Although we found that children who were presented with large numerosities successfully chose a target over a distractor quantity that differed by this 2:3 ratio (see Experiment 2a results), it remains possible that for some reason, children in our task struggled to discriminate a 2:3 ratio when presented with small number arrays.

To explore these issues, we conducted a third experiment in which children were asked to mentally transform small arrays, but this time their goal was to infer the value of an unknown sum rather than an unknown addend (e.g., $1 + 2 = x$; $x = 2$ or 3). Operations involving unknown sums or differences are often thought of as involving arithmetic operations, whereas operations involving unknown addends are often thought of as pre-algebraic (Nathan & Koedinger, 2000). Given that children much younger than those tested here can track successive additions of individual objects in quantity choice tasks (e.g., Feigenson, Carey, & Hauser, 2002; Feigenson, Carey, & Spelke, 2002), we expected children in Experiment 3 to succeed. If children in Experiment 3 succeed, then this would suggest that children's difficulty lies with inferring values of unknown addends from small number arrays, rather than in tracking small arrays at all. In addition, children in Experiment 3 saw target and distractor quantities that instantiated ratios that were equal to or harder to discriminate than those in Experiments 2a and 2b. If, as predicted, children succeeded in Experiment 3, this would rule out ratio effects as the source of children's failure with small arrays.

5. Experiment 3: Unknown sum, small quantities

5.1. Participants

Twenty-four children (mean age: 5 years, 3 months; range: 4 years 1 month – 6 years 11 months; 14 girls) participated at the science museum.

5.2. Methods

5.2.1. Materials

The materials were identical to those in Experiments 1 and 2a & b.

5.2.2. Procedure

Children first were introduced to Gator, as in our previous experiments. However, this time they were told that Gator had a "special cup" in which he always wanted to have a particular number of things, and that he could add things until he got that special number. Children were not told what this special number was, but were invited to infer it through a series of Demonstration trials (as in Experiments 1, 2a, and 2b). Children were randomly assigned to one of two conditions: the 2 + 1 Condition, in which Gator's special number was three, or the 1 + 1 Condition, in which Gator's special number was two. Including a condition in which the target number was three allowed us to test children with the more difficult ratio comparison of 3:4 at test, so that we could assess whether the failures in Experiments 2a and 2b were ratio-driven.

Children saw three Demonstration Trials. In the 2 + 1 condition, the experimenter held the cup up so that children could not see into it, and then placed one button on the table and said, "See this button? I'm going to put it in the cup!" She then placed the button inside the cup. The experimenter then placed two more buttons on the table, said, "See these buttons? I'm going to put them in cup!" and then placed the buttons in the cup. The experimenter then made Gator jump up and down and say "Yay! That's my special number!" The second and third Demonstration Trials proceeded similarly to the first; in the second Demonstration Trial, children saw two pennies + one penny, and in the third Demonstration Trial, children saw one shoe + two shoes. The 1 + 1 Condition proceeded identically to the 2 + 1 Condition, except that children saw one object followed by one more object added to the cup on each Demonstration Trial. Hence, just as in Experiments 1 and 2a & b, children saw two visible quantities and had to infer the value of a third quantity. In Experiment 3 the visible quantities were the two addends and the inferred quantity was their sum, whereas in Experiments 1 and 2a & b the visible quantities were the first addend and the sum, and the inferred quantity was the second addend.

In the single Test Trial, the experimenter placed two identical cups upside down on the empty table. The experimenter then said, "Look! I have two cups here, but only one of them is Gator's special cup. I'm going to show you what's inside, and you can tell me which one is Gator's special cup." She lifted both cups simultaneously to reveal two quantities of pom-poms, the target quantity and a distractor quantity. She then asked children "Which cup is Gator's?" In the 2 + 1 Condition, the target quantity was three and the distractor quantity was either two or four. In the 1 + 1 Condition, the target quantity was two and the distractor quantity was either one or three (see Table 1).

5.2.3. Quantities

As in Experiments 2a and 2b, the quantities used in the Demonstration trials were chosen to be within the range that is typically quantified using individual object representations. Test quantities were chosen to be within this same range, and also to be discriminable from the addend quantities using ratios that were as difficult or harder to discriminate than in Experiments 2a and b—this allowed us to ask whether ratio difficulty contributed to children's performance.

5.3. Results

When asked to choose which of two visible quantities was inferred to be in Gator's cup, 20/24 children correctly chose the target quantity (83.3% binomial test $p = 0.002$, two-tailed; Bayes factor = 63.16:1 *against* the null). There was no effect of Condition; 9/12 children correctly chose the target cup in the 2 + 1 condition, and 11/12 children correctly chose the target cup in the 1 + 1 condition; Fisher's exact test $p = 0.59$. There also was no effect of the quantity in the distractor cup: 11/14 children succeeded when the distractor quantity was larger than the target quantity, and 9/10 succeeded when the distractor quantity was smaller than the target quantity (Fisher's exact test $p = 0.61$). We also found no effect of gender (12/14 girls and 8/10 boys succeeded, Fisher's exact test $p = 1.0$) or age (one-way ANOVA $F_{1,22} = 2.01$, $p = 0.17$; Fisher-Freeman-Halton exact test $p = 0.79$). Finally, we compared children's performance in Experiment 3, in which they had to infer the value of an unknown sum, with their combined performance in Experiments 2a & b, in which they had to infer the value of an unknown addend. Children performed significantly better in Experiment 3 than in Experiments 2a & b (Fisher's exact test $p = 0.02$).

5.4. Discussion

In Experiment 3, children successfully inferred the value of an unknown sum when problems were presented non-symbolically with small numbers of objects. This success contrasts with children's performance in Experiments 2a & b, despite both experiments requiring children to compute over representations of individual objects. This divergence in results suggests that children's difficulty in Experiments 2a & b was due specifically to the challenges involved in solving for an unknown addend with small arrays, and not to drawing inferences over small object arrays more generally.

6. General discussion

Previous work (Kibbe & Feigenson, 2015) found that young children can "solve for x" in addend-unknown problems when the problems are presented non-symbolically in a way that encourages the use of the Approximate Number System (ANS). Here, we asked whether children also could solve for an unknown addend when problems were instead presented with smaller quantities that encouraged the use of individual object representations.

First, in Experiment 1, we replicated our earlier findings. We presented 4- to 6-year old children with non-symbolic addend-unknown problems, using large quantities of objects that could not be counted one by one, but that could be estimated using the ANS (e.g., $5 + x = 17$). As in our previous work, we found that children correctly chose which of two possible quantities was the value of the unknown addend (i.e., was the contents of "Gator's magic cup"). Next, in Experiment 2a, we tested another group of 4- to 6-year old children using the same study design, except this time with arrays of four or fewer objects, chosen to encourage the use of individual object representations rather than approximate number representations. We found that, unlike children in Experiment 1, children in Experiment 2a failed to identify which of two possible quantities was the unknown addend. We then replicated this surprising result with a new group of children in Experiment 2b. Children's failure was especially surprising since children of this age, who on average were over five years old, can count reliably in the small number range (e.g., LeCorre & Carey, 2007), and could potentially have solved the task by counting.

Finally, in Experiment 3, we showed that children's failures in Experiments 2a & b were not due to a lack of motivation to track small numbers of objects, or to difficulty discriminating between the two quantities presented at test. When children were shown transformations of the same small quantities as in Experiments 2a & b, but were asked to solve for an unknown sum rather than an unknown addend, they succeeded. This success suggests that children's failures in Experiments 2a & b were specific to solving for an unknown addend using individual object representations.

Children's success at performing pre-algebraic computations using ANS representations, and their failure to do so using individual object representations, is surprising given that even very young (pre-verbal) children can perform quantity-relevant computations that using either representational system (e.g., $1 + 1 = 2$, Wynn, 1992; $5 + 5 = 10$, McCrink & Wynn, 2004). However, our results suggest that there may be important differences in the kinds of computations that children spontaneously perform with these different types of representations. Whereas some computations, such as addition, may be easily supported by both types of representation, other computations, such as solving for an unknown addend, may be supported by ANS representations but not representations of individual objects. One intriguing possibility is that solving for an unknown *x*, as when determining the value of an unknown addend, requires mentally representing a variable. The variable is represented as having some cardinal value (even if that value is currently unknown). Representations of approximate number seem well suited to play such a role—approximate number is a type of summary property bound to a representation of a single ensemble entity (Halberda, Sires, & Feigenson, 2006). As such, approximate number representations are inherently cardinal in nature—they carry information about the array or the collection as a whole, rather than about any one of its constituents. One could know that any given collection must have *some* cardinality—some approximate numerosity—even without knowing anything about the size of that value. In contrast, representations of individual objects may not be not well suited to serve as the values of variables, because they have no inherent cardinality. If one asks "How many?", representations of individual objects (Object A, Object B, Object C) are not a good answer to the question, just as representations of individual people (Bob, Mary, Jim) are not a good answer. Rather, one needs a higher order representation that unifies across the individuals—a representation of cardinality. It is possible to represent individual objects as forming a single entity, a set, that has a cardinal value (Feigenson, 2011), but this requires two extra computations in addition to representing the individual objects—one has to bind the individual object representations into a set, and then compute the set's approximate or exact cardinality (Halberda & Feigenson, 2008b). These computations seem not to have been performed by the children in our task.

Why else might representations of small number fail to support solving for *x*? Another potential source of difficulty is that, when presented with small quantities, children often preferentially encode the individual identities of the objects (e.g. Cantrell & Smith, 2013; Feigenson, Carey, & Hauser, 2002). In our task, children were shown a variety of objects whose identities could differ both within and between trials. Children presented with small quantities may have found the shape or color of each individual object to be more salient than the array as a whole, and may thus have failed to attend to the cardinalities across Demonstration Trials. For children who were presented with large quantities, in contrast, the variety of object identities across trials may have helped children generate consistent, abstract ANS representations. Children's success in Experiment 3 suggests that, under some circumstances, children can attend to quantity when presented with small quantities of varieties of objects. However, the salience of the identities of individual objects may be one factor contributing to children's difficulty in the more challenging Experiments 2a & b. The salience or relevance of identity when representing large collections of items using the ANS, versus when representing small arrays using individual object representations, remains an important topic for future work.

We also note that children's ability to "solve for x" with small quantities may not be all or none. Instead, children's ability to solve for unknowns may be graded—with small quantities posing more difficulty than large quantities, but still supporting partial success. A hint of evidence for this comes from Experiments 2a and 2b, in which children showed more success when the distractor quantity contained only a single object (1 vs. 2 trials) than when it contained more objects (2 vs. 3 trials). It seems possible that when children are shown very small quantities, or quantities that differ by a large enough ratio (1 vs. 2, as opposed to 2 vs. 3), they could succeed at solving for x. However, it is also possible that this partial success was due to a simple bias to choose the larger of the two quantities when they children were uncertain, a possibility that is supported by children's tendency to choose the distractor when it is larger than the target (2 vs. 3). Future work could adjudicate between these possibilities.

In summary, here we report a surprising dissociation in children's numerical performance. Four- to 6-year old children, who have received little or no formal mathematics education, successfully solved pre-algebraic problems when those problems were presented non-symbolically with arrays of objects rather than with digits or number words. However, children did not solve these addend unknown problems with equal ease across quantities. Rather, they succeeded with large approximate quantities, and failed with small exact quantities. This might seem counter to the intuition that mathematical computations are easier for children when they involve small, countable arrays. Although further work will be required to more closely examine this finding, our results suggest that encouraging children to draw upon their representations of approximate numerosities may nudge them towards intuitively performing the relevant mathematical computation (here, "back-solving" for an unknown addend), which could later help them as they encounter problems that require an exact solution.

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Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.cognition.2016.12.006.

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